Math 2568 Eigenvalues & Eigenvectors Key

1. $\begin{vmatrix} 1-\lambda & 6 \\ 2 & 5-\lambda \end{vmatrix} = (1-\lambda)(5-\lambda) - 12 = 5 - \lambda - 5\lambda + \lambda^2 - 12 =$

$\lambda^2 - 6\lambda - 7 = 0 \Rightarrow (\lambda - 7)(\lambda + 1) = 0 \quad \lambda = -1, 7$

$\lambda_1 = -1$ $\begin{bmatrix} 1+1 & 6 \\ 2 & 6 \end{bmatrix}$ $\Downarrow$ $\begin{bmatrix} 2 & 6 \\ 0 & 0 \end{bmatrix}$

$2x_1 + 6x_2 = 0$
$x_1 + 3x_2 = 0$
$x_1 = -3x_2$
$x_2 = x_2.$

$\vec{v}_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$\lambda_2 = 7$ $\begin{bmatrix} -6 & 6 \\ 2 & -2 \end{bmatrix}$ $\Downarrow$ $\begin{bmatrix} -6 & 6 \\ 0 & 0 \end{bmatrix}$

$-6x_1 + 6x_2 = 0$
$-x_1 + x_2 = 0$
$x_1 = x_2$
$x_2 = x_2$

$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2. $\begin{vmatrix} 2-\lambda & 9 \\ 1 & 10-\lambda \end{vmatrix} = (2-\lambda)(10-\lambda) - 9 = 20 - 2\lambda - 10\lambda + \lambda^2 - 9 =$

$\lambda^2 - 12\lambda + 11 = 0 \Rightarrow (\lambda - 11)(\lambda - 1) = 0$

$\lambda = 11, 1$

$\lambda_1 = 11$ $\begin{bmatrix} -9 & 9 \\ 1 & -1 \end{bmatrix}$ $\Downarrow$ $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$x_1 - x_2 = 0$
$x_1 = x_2$
$x_2 = x_2$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 1$ $\begin{bmatrix} 1 & 9 \\ 1 & 9 \end{bmatrix}$ $\Downarrow$ $\begin{bmatrix} 1 & 9 \\ 0 & 0 \end{bmatrix}$

$x_1 + 9x_2 = 0$
$x_1 = -9x_2$
$x_2 = x_2$

$\vec{v}_2 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$

3. $\begin{vmatrix} 3-\lambda & 4 \\ 0 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) - 0 = 0 \Rightarrow \lambda = 3, \lambda = 2$

$\lambda_1 = 3$ $\begin{bmatrix} 0 & 4 \\ 0 & 1 \end{bmatrix} \Downarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $4x_2 = 0$, $x_2 = 0$, $x_1 = \text{free} = x_1$ $\quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_2 = 2$ $\begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}$ $x_1 + 4x_2 = 0$, $x_1 = -4x_2$, $x_2 = x_2$ $\quad \vec{v}_2 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

4. $\begin{vmatrix} 2-\lambda & 0 \\ 1 & 3-\lambda \end{vmatrix}$ as above matrix is triangular so eigenvalues are diagonal entries $\lambda = 2, \lambda = 3$

$\lambda_1 = 2$ $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ $x_1 + x_2 = 0$, $x_1 = -x_2$, $x_2 = x_2$ $\quad \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = 3$ $\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \Downarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $x_1 = 0$, $x_2 = x_2$ $\quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

5. $\begin{vmatrix} 4-\lambda & 5 \\ 6 & 11-\lambda \end{vmatrix} = (4-\lambda)(11-\lambda) - 30 = 44 - 4\lambda - 11\lambda + \lambda^2 - 30 =$

$\lambda^2 - 15\lambda + 14 = 0 \Rightarrow (\lambda-1)(\lambda-14) = 0 \quad \lambda = 1, 14$

$\lambda_1 = 1$ $\begin{bmatrix} 3 & 5 \\ 6 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$ $3x_1 + 5x_2 = 0$, $x_1 = -5/3\, x_2$, $x_2 = x_2$ $\quad \vec{v}_1 = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$

5 cont'd

$\lambda_2 = 14 \quad \begin{bmatrix} -10 & 5 \\ 6 & -3 \end{bmatrix} \Downarrow \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$

$2x_1 - x_2 = 0$

$x_1 = \frac{1}{2} x_2$

$x_2 = x_2$

$\vec{V_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

6. $\begin{vmatrix} -\lambda & 1 \\ 8 & 2-\lambda \end{vmatrix} = (-\lambda)(2-\lambda) - 8 = -2\lambda + \lambda^2 - 8 = \lambda^2 - 2\lambda - 8 = 0$

$(\lambda - 4)(\lambda + 2) = 0 \quad \lambda = 4, -2$

$\lambda_1 = 4 \quad \begin{bmatrix} -4 & 1 \\ 8 & -2 \end{bmatrix} \Downarrow \begin{bmatrix} -4 & 1 \\ 0 & 0 \end{bmatrix}$

$-4x_1 + x_2 = 0$

$x_1 = \frac{1}{4} x_2$

$x_2 = x_2$

$\vec{V_1} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$\lambda_2 = -2 \quad \begin{bmatrix} 2 & 1 \\ 8 & 4 \end{bmatrix} \Downarrow \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$

$2x_1 + x_2 = 0$

$x_1 = -\frac{1}{2} x_2$

$x_2 = x_2$

$\vec{V_2} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

7. $\begin{vmatrix} -1-\lambda & 1 \\ 3 & 1-\lambda \end{vmatrix} = (-1-\lambda)(1-\lambda) - 3 = -1 + \lambda - \lambda + \lambda^2 - 3 = \lambda^2 - 4 = 0 \Rightarrow$

$(\lambda - 2)(\lambda + 2) = 0 \quad \lambda = 2, -2$

$\lambda_1 = 2 \quad \begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix}$

$-3x_1 + x_2 = 0$

$x_1 = \frac{1}{3} x_2$

$x_2 = x_2$

$\vec{V_1} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\lambda_2 = -2 \quad \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$x_1 + x_2 = 0$

$x_1 = -x_2$

$x_2 = x_2$

$\vec{V_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

8. $\begin{vmatrix} -2-\lambda & 5 \\ 7 & -\lambda \end{vmatrix} = (-2-\lambda)(-\lambda) - 35 = 2\lambda + \lambda^2 - 35 = \lambda^2 + 2\lambda - 35 = 0$

$\Rightarrow (\lambda+7)(\lambda-5) = 0 \qquad \lambda = -7, 5$

$\lambda_1 = -7 \quad \begin{bmatrix} 5 & 5 \\ 7 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{aligned} x_1 + x_2 &= 0 \\ x_1 &= -x_2 \\ x_2 &= x_2 \end{aligned} \quad \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = 5 \quad \begin{bmatrix} -7 & 5 \\ 7 & -5 \end{bmatrix} \Rightarrow \begin{bmatrix} -7 & 5 \\ 0 & 0 \end{bmatrix} \quad \begin{aligned} -7x_1 + 5x_2 &= 0 \\ x_1 &= \tfrac{5}{7}x_2 \\ x_2 &= x_2 \end{aligned} \quad \vec{v}_2 = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

9. $\begin{vmatrix} -3-\lambda & 7 \\ 5 & -1-\lambda \end{vmatrix} = (-3-\lambda)(-1-\lambda) - 35 = (3+\lambda)(1+\lambda) - 35 =$

$3 + 3\lambda + \lambda + \lambda^2 - 35 = \lambda^2 + 4\lambda - 32 = 0 \Rightarrow \lambda = -8, 4$

$(\lambda+8)(\lambda-4)$

$\lambda_1 = -8 \quad \begin{bmatrix} 5 & 7 \\ 5 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 & 7 \\ 0 & 0 \end{bmatrix} \quad \begin{aligned} 5x_1 + 7x_2 &= 0 \\ x_1 &= -\tfrac{7}{5}x_2 \\ x_2 &= x_2 \end{aligned} \quad \vec{v}_1 = \begin{bmatrix} -7 \\ 5 \end{bmatrix}$

$\lambda_2 = 4 \quad \begin{bmatrix} -7 & 7 \\ 5 & -5 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{aligned} -x_1 + x_2 &= 0 \\ x_1 &= x_2 \\ x_2 &= x_2 \end{aligned} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

10. $\begin{vmatrix} -4-\lambda & 1 \\ 6 & -5-\lambda \end{vmatrix} = (4+\lambda)(5+\lambda) - 6 = 20 + 9\lambda + \lambda^2 - 6 = \lambda^2 + 9\lambda + 14 = 0$

$(\lambda+7)(\lambda+2) = 0 \Rightarrow \lambda = -7, -2$

$\lambda_1 = -7 \quad \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{aligned} 3x_1 + x_2 &= 0 \\ x_1 &= -\tfrac{1}{3}x_2 \\ x_2 &= x_2 \end{aligned} \quad \vec{v}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$\lambda_2 = -2 \quad \begin{bmatrix} -2 & 1 \\ 6 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{aligned} -2x_1 + x_2 &= 0 \\ x_1 &= \tfrac{1}{2}x_2 \\ x_2 &= x_2 \end{aligned} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

11. $\begin{vmatrix} 3-\lambda & -2 \\ 2 & 3-\lambda \end{vmatrix} = (3-\lambda)(3-\lambda)+4 = 9-6\lambda+\lambda^2+4 =$

$\lambda^2-6\lambda+13=0 \quad \lambda = \frac{6 \pm \sqrt{36-52}}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i$

$\lambda_1 = 3+2i$

$$\begin{bmatrix} 3-(3+2i) & -2 \\ 2 & 3-(3+2i) \end{bmatrix} = \begin{bmatrix} 3-3-2i & -2 \\ 2 & 3-3-2i \end{bmatrix} = \begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix} = \begin{bmatrix} 2 & -2i \\ 0 & 0 \end{bmatrix}$$

$2x_1 - 2ix_2 = 0 \Rightarrow \frac{2x_1}{2} = \frac{(2i)x_2}{2} \Longrightarrow \begin{matrix} x_1 = i\,x_2 \\ x_2 = x_2 \end{matrix} \quad \vec{v_1} = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i$

$\lambda_2 = 3-2i \quad \begin{bmatrix} 3-(3-2i) & -2 \\ 2 & 3-(3-2i) \end{bmatrix} = \begin{bmatrix} 3-3+2i & -2 \\ 2 & 3-3+2i \end{bmatrix} = \begin{bmatrix} 2i & -2 \\ 2 & 2i \end{bmatrix}$

$= \begin{bmatrix} 2 & 2i \\ 0 & 0 \end{bmatrix} \quad \begin{matrix} 2x_1 + (2i)x_2 = 0 \\ x_1 = -i\,x_2 \\ x_2 = x_2 \end{matrix} \quad \vec{V_2} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} i$

notice that $\vec{V_2}$ is the complex conjugate of $\vec{v_1}$ this is always true

12. $\begin{vmatrix} -4-\lambda & 5 \\ -5 & -4-\lambda \end{vmatrix} = (4+\lambda)(4+\lambda)+25 = \lambda^2+8\lambda+41=0$

$\lambda = \frac{-8 \pm \sqrt{64-164}}{2} = \frac{-8 \pm 10i}{2} = -4 \pm 5i$

$\lambda_1 = -4+5i$

$\begin{bmatrix} -5i & 5 \\ -5 & -5i \end{bmatrix} = \begin{bmatrix} -5 & -5i \\ 0 & 0 \end{bmatrix} \quad \begin{matrix} -5x_1 - 5ix_2 = 0 \\ x_1 = -ix_2 \\ x_2 = x_2 \end{matrix} \quad \vec{v_1} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} i$

$\lambda_2 = -4-5i \quad \Rightarrow \vec{v_2} = \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i$

13. $\begin{vmatrix} -2-\lambda & 2 \\ -5 & 6-\lambda \end{vmatrix} = (-2-\lambda)(6-\lambda)+10 = -12+2\lambda-6\lambda+\lambda^2+10 = \lambda^2-4\lambda-2=0$

$\lambda = \frac{4 \pm \sqrt{16+8}}{2} = \frac{4 \pm \sqrt{24}}{2} = \frac{4 \pm 2\sqrt{6}}{2} = 2 \pm \sqrt{6}$

13. Cont'd

$\lambda_1 = 2+\sqrt{6}$ $\begin{bmatrix} -2-(2+\sqrt{6}) & 2 \\ -5 & 6-(2+\sqrt{6}) \end{bmatrix} = \begin{bmatrix} -2-2-\sqrt{6} & 2 \\ -5 & 4-\sqrt{6} \end{bmatrix} = \begin{bmatrix} -4-\sqrt{6} & 2 \\ -5 & 4-\sqrt{6} \end{bmatrix}$

$= \begin{bmatrix} -5 & 4-\sqrt{6} \\ 0 & 0 \end{bmatrix}$ $-5x_1 + (4-\sqrt{6})x_2 = 0$

$x_1 = \dfrac{4-\sqrt{6}}{5}x_2$ $\quad \vec{v}_1 = \begin{bmatrix} 4-\sqrt{6} \\ 5 \end{bmatrix}$

$x_2 = x_2$

$\lambda_2 = 2-\sqrt{6}$ $\begin{bmatrix} -2-(2-\sqrt{6}) & 2 \\ -5 & 6-(2-\sqrt{6}) \end{bmatrix} = \begin{bmatrix} -2-2+\sqrt{6} & 2 \\ -5 & 6-2+\sqrt{6} \end{bmatrix} = \begin{bmatrix} -4+\sqrt{6} & 2 \\ -5 & 4+\sqrt{6} \end{bmatrix}$

$= \begin{bmatrix} -5 & 4+\sqrt{6} \\ 0 & 0 \end{bmatrix}$ $-5x_1 + (4+\sqrt{6})x_2 = 0$

$x_1 = \dfrac{4+\sqrt{6}}{5}x_2$ $\quad \vec{V}_2 = \begin{bmatrix} 4+\sqrt{6} \\ 5 \end{bmatrix}$

$x_2 = x_2$

also note that irrational roots also produce conjugate eigenvectors

14. $\begin{vmatrix} -2-\lambda & 5 & 3 \\ 0 & 2-\lambda & -4 \\ 0 & -1 & 2-\lambda \end{vmatrix} = (-2-\lambda)\left[(2-\lambda)(2-\lambda)-4\right] = 0 \Rightarrow$

$-(2+\lambda)\left[\cancel{4}-4\lambda+\lambda^2-\cancel{4}\right] = 0 \Rightarrow$

$(2+\lambda)\left[\lambda^2-4\lambda\right] = 0 \Rightarrow \lambda = -2, 0, 4$

$\lambda(\lambda-4)$

$\lambda_1 = -2$ $\begin{bmatrix} 0 & 5 & 3 \\ 0 & 4 & -4 \\ 0 & -1 & 4 \end{bmatrix}$ rref $\Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_1 = \text{free} = x_1$, $x_2 = 0$, $x_3 = 0$ $\quad \vec{V}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_2 = 0$ $\begin{bmatrix} -2 & 5 & 3 \\ 0 & 2 & -4 \\ 0 & -1 & 2 \end{bmatrix}$ rref $\Rightarrow \begin{bmatrix} 1 & 0 & -13/2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_1 - 13/2\, x_3 = 0$, $x_2 - 2x_3 = 0 \Rightarrow$

$x_1 = \frac{13}{2}x_3$

$x_2 = 2x_3$ $\quad \vec{V}_2 = \begin{bmatrix} 13 \\ 4 \\ 2 \end{bmatrix}$

$x_3 = x_3$

$\lambda_3 = 4$ $\begin{bmatrix} -6 & 5 & 3 \\ 0 & -2 & -4 \\ 0 & -1 & -2 \end{bmatrix}$ rref $\Rightarrow \begin{bmatrix} 1 & 0 & 7/6 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 = -7/6\, x_3$

$x_2 = -2x_3$ $\quad \vec{V}_3 = \begin{bmatrix} -7 \\ -12 \\ 6 \end{bmatrix}$

$x_3 = x_3$

15. $\begin{vmatrix} 4-\lambda & -2 & 0 \\ 0 & -\lambda & -3 \\ 0 & 2 & 5-\lambda \end{vmatrix} = (4-\lambda)[(-\lambda)(5-\lambda)+6] = (4-\lambda)[-5\lambda+\lambda^2+6] =$

$(4-\lambda)(\lambda^2-5\lambda+6) = (4-\lambda)(\lambda-2)(\lambda-3) = 0$

$\Rightarrow \lambda = 4, 2, 3$

$\lambda_1 = 4$ $\begin{bmatrix} 0 & -2 & 0 \\ 0 & -4 & -3 \\ 0 & 2 & 1 \end{bmatrix}$ rref $\Rightarrow$ $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{aligned} x_1 &= \text{free} = x_1 \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned}$ $\vec{V_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_2 = 2$ $\begin{bmatrix} 2 & -2 & 0 \\ 0 & -2 & -3 \\ 0 & 2 & 3 \end{bmatrix}$ rref $\Rightarrow$ $\begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{aligned} x_1 &= -3/2 x_3 \\ x_2 &= -3/2 x_3 \\ x_3 &= x_3 \end{aligned}$ $\vec{V_2} = \begin{bmatrix} -3 \\ -3 \\ 2 \end{bmatrix}$

$\lambda_3 = 3$ $\begin{bmatrix} 1 & -2 & 0 \\ 0 & -3 & -3 \\ 0 & 2 & 2 \end{bmatrix}$ rref $\Rightarrow$ $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{aligned} x_1 &= -2x_3 \\ x_2 &= -x_3 \\ x_3 &= x_3 \end{aligned}$ $\vec{V_3} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$