MTH 291, Final Exam, Fall 2017 Name KEY

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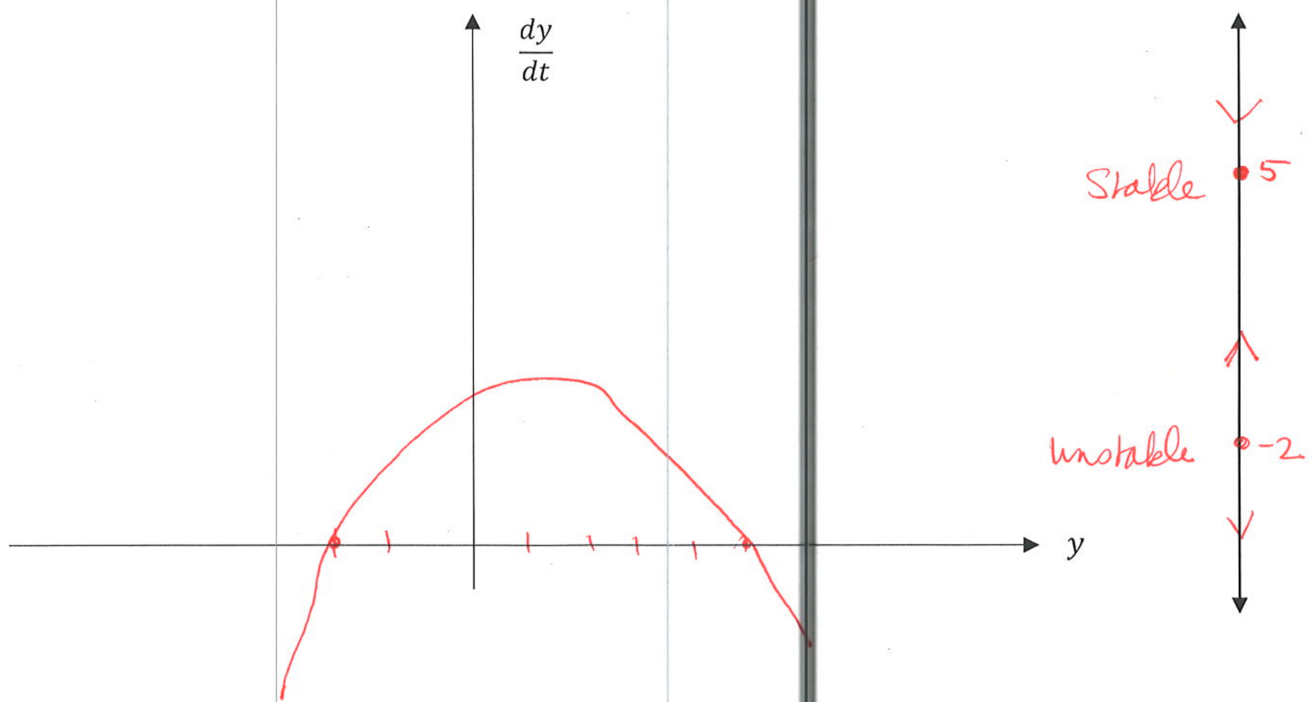
1. Sketch the phase plane of $\frac{dy}{dt} = 10 + 3y - y^2$ on the axis below, and then convert that to a phase line. Describe the stability of each point. (15 points) $-(y^2-3y-10) = -(y-5)(y+2)$



2. Solve the differential equation $\frac{dy}{dx} = x\sqrt{1-y^2}$. (15 points)

$$\frac{dy}{\sqrt{1-y^2}} = x\,dx$$

$$\arcsin y = \frac{1}{2}x^2 + C$$

$$y = \sin\left(\frac{x^2}{2} + C\right)$$

3. Solve the differential equation $\frac{dy}{dx} = 2y + x^2 + 5$ by the method of integrating factors (reverse product rule). (15 points)

$\mu = e^{\int -2dx} = e^{-2x}$

$\frac{dy}{dx} - 2y = x^2 + 5$

$e^{-2x}y' - 2e^{-2x}y = (x^2+5)e^{-2x}$

$\int (e^{-2x}y)' = \int (x^2+5)e^{-2x} + C$

$e^{-2x}y = -\frac{1}{4}e^{-2x}(2x^2+2x+11) + C$

$y = -\frac{1}{4}(2x^2+2x+11) + Ce^{2x}$

4. Solve the second order differential equation $3y'' + 2y' + y = 0$ for the general solution. (15 points)

$3r^2 + 2r + 1 = 0$

$r = \frac{-2 \pm \sqrt{4-12}}{2(3)} = \frac{-2 \pm \sqrt{8}i}{6} = -\frac{1}{3} \pm \frac{\sqrt{2}}{3}i$

$2\sqrt{2}$

$y_1 = c_1 e^{-1/3t}\cos(\frac{\sqrt{2}}{3}t) + c_2 e^{-1/3t}\sin(\frac{\sqrt{2}}{3}t)$

5. Solve the second order differential equation $y'' - 10y' + 25y = 0$ for the general solution. (15 points)

$r^2 - 10r + 25 = 0$

$(r-5)^2 = 0$

$r = 5$

$y = c_1 e^{5t} + c_2 t e^{5t}$

6. Identify the Ansatz to find the particular solution for the differential equation $y'' + 4y = f(x)$. (5 points each)

$y_1 = \cos 2x$
$y_2 = \sin 2x$

a. $f(x) = x^2 - 2x$

$Y(x) = Ax^2 + Bx + C$

b. $f(x) = 2e^{4x}$

$Y(x) = Ae^{4x}$

c. $f(x) = 3\sin 2x$

$Y(x) = Ax\cos 2x + Bx\sin 2x$

7. Find the general solutions to the systems. (15 points each)

a. $\vec{x}' = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \vec{x}$

$(2-\lambda)(-1-\lambda) - 1 = 0$

$\lambda^2 - \lambda - 2 - 1 = 0$

$\lambda^2 - \lambda - 3 = 0$

$\lambda = \frac{1 \pm \sqrt{1+12}}{2}$

$= \frac{1}{2} \pm \frac{\sqrt{13}}{2}$

$\begin{bmatrix} 2 - (\frac{1}{2} + \frac{\sqrt{13}}{2}) & 1 \\ 1 & -1 - (\frac{1}{2} + \frac{\sqrt{13}}{2}) \end{bmatrix} =$

$\begin{bmatrix} 3/2 - \frac{\sqrt{13}}{2} & 1 \\ 1 & -3/2 - \frac{\sqrt{13}}{2} \end{bmatrix}$

$x_1 - (3/2 + \frac{\sqrt{13}}{2}) x_2 = 0$

$x_1 = \left(\frac{3}{2} + \frac{\sqrt{13}}{2}\right) x_2$

$\vec{v}_1 = \begin{bmatrix} 3 + \sqrt{13} \\ 2 \end{bmatrix}$ $\quad \vec{v}_2 = \begin{bmatrix} 3 - \sqrt{13} \\ 2 \end{bmatrix}$

$\vec{X} = c_1 \begin{bmatrix} 3 + \sqrt{13} \\ 2 \end{bmatrix} e^{\left(\frac{1+\sqrt{13}}{2}\right)t} + c_2 \begin{bmatrix} 3 - \sqrt{13} \\ 2 \end{bmatrix} e^{\left(\frac{1-\sqrt{13}}{2}\right)t}$

b. $\vec{x}' = \begin{bmatrix}1 & -8\\1 & -3\end{bmatrix}\vec{x}$

$(1-\lambda)(-3-\lambda)+8=0$

$\lambda^2+2\lambda-3+8=0$

$\lambda^2+2\lambda+5=0$

$\lambda = \frac{-2 \pm \sqrt{4-20}}{2}$

$= -1 \pm 2i$

$\begin{bmatrix} 1-(-1+2i) & -8 \\ 1 & -3-(-1+2i) \end{bmatrix}$

$\begin{bmatrix} 2-2i & -8 \\ 1 & -2-2i \end{bmatrix}$

$x_1 - (2+2i)x_2 = 0$

$x_1 = (2+2i)x_2$

$\vec{v}_1 = \begin{bmatrix} 2+2i \\ 1 \end{bmatrix}$

$e^{-t}\begin{bmatrix} 2+2i \\ 1 \end{bmatrix}(\cos 2t + i\sin 2t) =$

$e^{-t}\begin{bmatrix} 2\cos 2t + 2i\sin 2t + 2i\cos 2t - 2\sin 2t \\ \cos 2t + i\sin 2t \end{bmatrix}$

$\vec{x} = c_1\begin{bmatrix} 2\cos 2t - 2\sin 2t \\ \cos 2t \end{bmatrix}e^{-t} + c_2\begin{bmatrix} 2\sin 2t + 2\cos 2t \\ \sin 2t \end{bmatrix}e^{-t}$

8. Use Euler's method for two steps by hand to estimate the value of $y(1.5)$ if $y' = 2x - 3y + 1$ and $y(1) = 5$. (15 points)

$\Delta t = .5/2 = .25$

$n=0$ $x_0 = 1$ $y_0 = 5$ $m_0 = 2(1) - 3(5) + 1 = -12$ $\Delta t = .25$ $y_1 = .25(-12)+5$
$= -3+5 = 2$

$n=1$ $x_1 = 1.25$ $y_1 = 2$ $m_1 = 2(1.25) - 3(2) + 1 = -2.5$

$y_2 = .25(-2.5) + 2$
$= 1.375$

$n=2$ $x_2 = 1.5$ $y_2 = 1.375$

$y(1.5) \approx 1.375$

MTH 291, Final Exam Supplement, Fall 2017   Name KEY

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1. Consider the differential equation $y' = y^2 + 4$. (4 points each)
   a. Explain why there are no constant solutions of the differential equation.

   there is no real solutions to $y' = 0$ / $0 = y^2 + 4$

   b. Describe the graph of the solution $y(x)$ (for example, can a solution to the curve have any relative extrema?).

   there can be no extrema, since y' can never be zero, but there will be an inflection point at $y = 0$

   c. Explain why $y = 0$ is the y-coordinate of a point of inflection on the solution curve.

   $y' = y^2 + 4$

   implicit differentiation

   $y'' = \frac{d^2y}{dt^2} = 2y\frac{dy}{dt} = 2y(y^2+4) = 0$

   $y = 0$ is a solution