

Instructions: Show all work. You will earn full credit for correct answers only when accompanied by work or explanation. Answers that are incorrect and have no work will not receive any partial credit. Use exact answers, except in applied problems: round to two decimal places, or the number requested in the problem.

1. Verify that $y = (1 - \sin x)^{-1/2}$ is a solution to the differential equation $2y' = y^3 \cos x$. (10 points)

$$y' = -\frac{1}{2}(1 - \sin x)^{-3/2}(-\cos x) = \frac{1}{2}(1 - \sin x)^{-3/2} \cos x$$

$$2y' = (1 - \sin x)^{-3/2} \cos x$$

$$y^3 \cos x = [(1 - \sin x)^{-1/2}]^3 \cos x = (1 - \sin x)^{-3/2} \cos x$$

they are equal, so it is a solution.

2. Verify that the pair of function $x = e^{-2t} + 3e^{6t}$ and $y = -e^{-2t} + 5e^{6t}$ is a set of solutions to the system $\frac{dx}{dt} = x + 3y$ and $\frac{dy}{dt} = 5x + 3y$. (12 points)

$$\frac{dx}{dt} = -2e^{-2t} + 18e^{6t} \quad x + 3y = e^{-2t} + 3e^{6t} + 3(-e^{-2t} + 5e^{6t})$$

$$= e^{-2t} + 3e^{6t} - 3e^{-2t} + 15e^{6t}$$

$$= -2e^{-2t} + 18e^{6t}$$

$$\frac{dy}{dt} = 2e^{-2t} + 30e^{6t} \quad 5x + 3y = 5(e^{-2t} + 3e^{6t}) + 3(-e^{-2t} + 5e^{6t})$$

$$= 5e^{-2t} + 15e^{6t} - 3e^{-2t} + 15e^{6t}$$

$$= 2e^{-2t} + 30e^{6t}$$

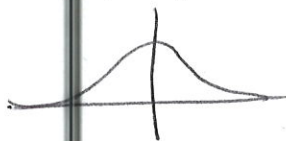
it is a solution

3. Consider the differential equation $\frac{dy}{dx} = e^{-x^2}$. Explain why a solution of the differential equation must be an increasing function on any interval of the x-axis. (10 points)

e^{-x^2} is always positive

So any solution to this

ODE must have a positive derivative, i.e. be increasing everywhere



4. Write a differential equation to model a situation where a population growth is proportional to the square of the population. (8 points)

$$\frac{dP}{dt} = kP^2$$

5. Suppose that a large mixing tank initially holds 300 gallons of water, into which is dissolved 50 pounds of salt. Another brine solution is pumped into the tank at a rate of 3 gal/min, and when the solution is well-stirred, it is then pumped out at a slower rate of 2 gal/min.
- a. If the concentration of the solution entering is 2 lbs/gal, determine a differential equation for the amount of salt at time t . (10 points)

$$\frac{dS}{dt} = \frac{3 \text{ gal}}{\text{min}} \cdot \frac{2 \text{ lbs}}{\text{gal}} - \frac{2 \text{ gal}}{\text{min}} \frac{S}{300+t \text{ gal}}$$

$$S(0) = 50$$

$$\frac{dS}{dt} = 6 - \frac{2S}{300+t}$$

- b. Solve the equation you found. How long is the equation defined? Assume the tank holds 1000 gallons of water. (8 points)

$$\frac{dS}{dt} + \frac{2}{300+t} S = 6$$

$\leftarrow t = 700 \text{ minutes}$

$$\mu = e^{\int \frac{2}{300+t} dt} = e^{2 \ln(300+t)} = e^{\ln(300+t)^2} = (300+t)^2$$

$$(300+t)^2 \frac{dS}{dt} + 2(300+t)S = 6(300+t)^2$$

$$S = 2(300+t) + \frac{C}{(300+t)^2}$$

$$\int [(300+t)^2 S]' = \int 6(300+t)^2 dt = 2(300+t)^3 + C$$

- c. If the solution is defined for all positive time, find the limit of the concentration. If the equation predicts the tank overflows, find the concentration at that time. (8 points)

$$50 = 600 + 2(0) + \frac{C}{(300+0)^2}$$

$$50 = 600 + \frac{C}{90,000}$$

$$-550 = \frac{C}{90,000}$$

$$C = -4.95 \times 10^7$$

$$S = 600 + 2t - \frac{4.95 \times 10^7}{(300+t)^2}$$

tank overflows at 700 minutes

$$S = 2(300+700) - \frac{4.95 \times 10^7}{(300+700)^2}$$

$$S = 1950.5$$

$$\frac{S}{\text{gal}} = \frac{1950.5}{1000} = 1.95 \text{ g/gal.}$$

6. Identify the slope fields that match each differential equation (4 points each)

i. $\frac{dy}{dx} = x^2 - y^2$

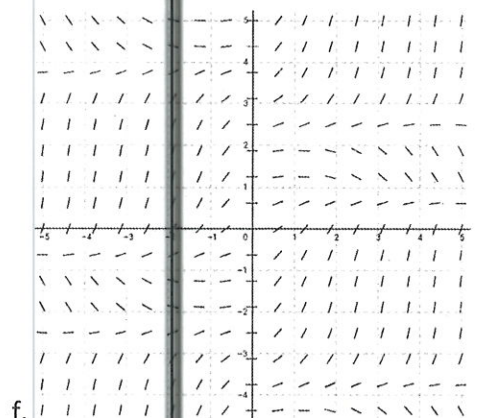
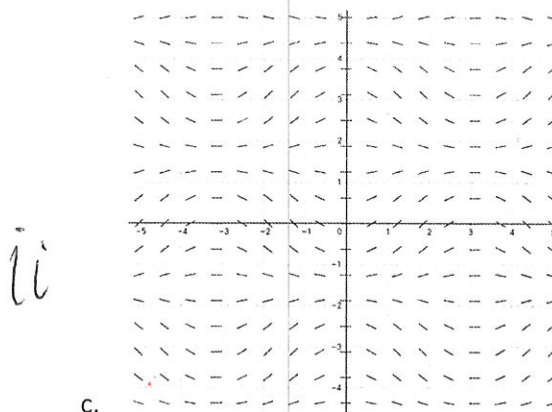
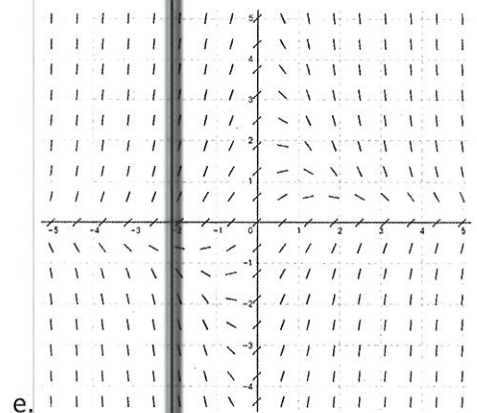
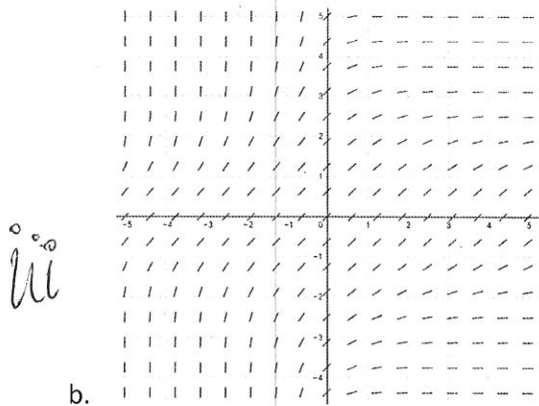
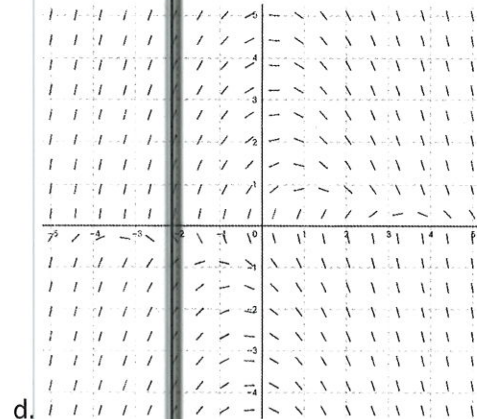
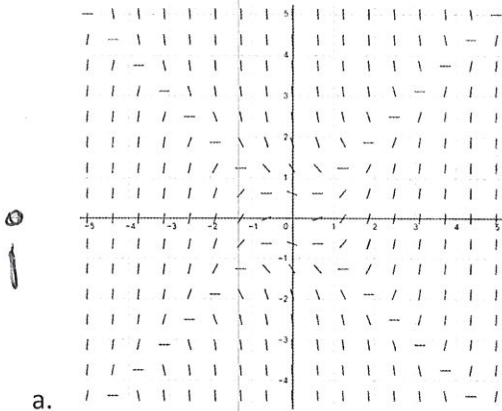
iii. $\frac{dy}{dx} = e^{-0.1xy^2}$

v. $\frac{dy}{dx} = 1 - xy$

ii. $\frac{dy}{dx} = (\sin x)(\cos x)$

iv. $\frac{dy}{dx} = \frac{1}{y} - x$

vi. $\frac{dy}{dx} = x^{2/3} - x \sin y$

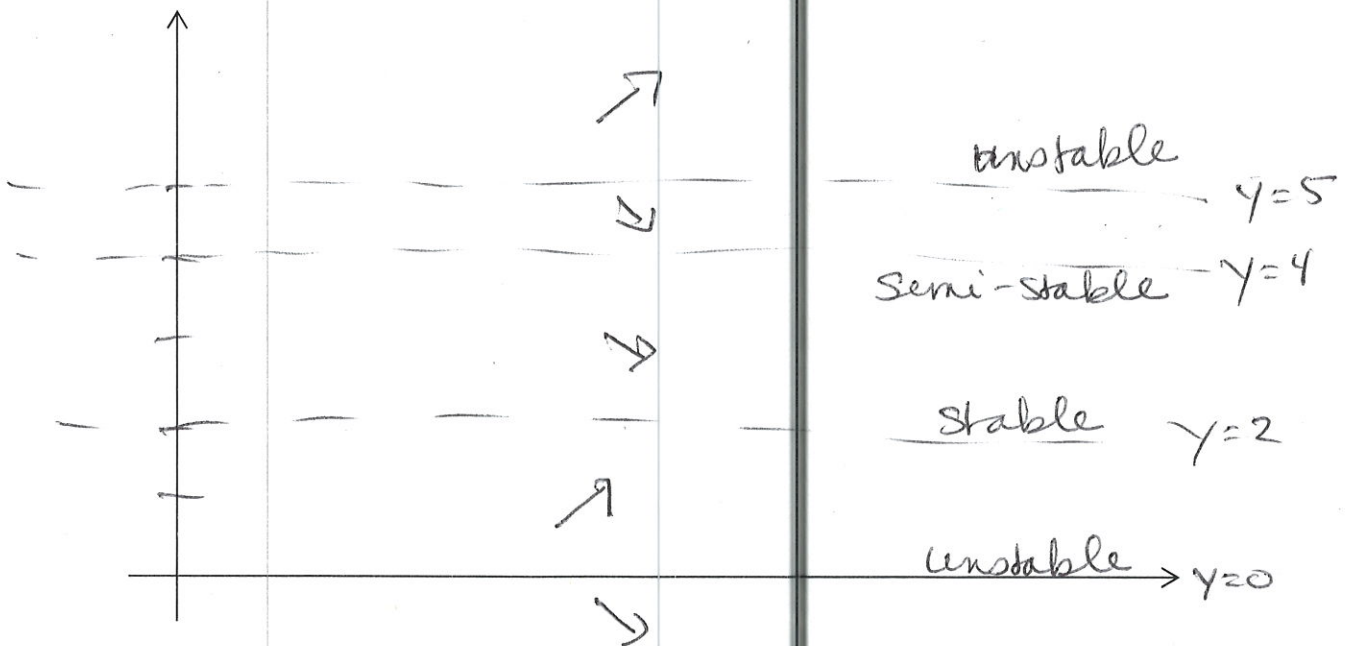
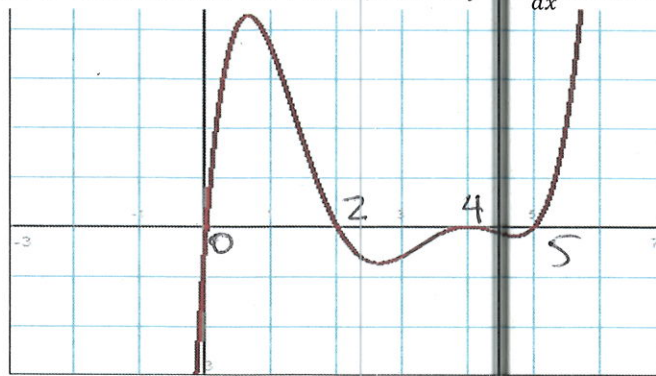


iv

v

vi

7. a. Sketch the slope field consistent with the phase plane of y vs. $\frac{dy}{dx}$ shown below. (8 points)



b. What would the phase line look like? (7 points)

